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General solution and invariants for a class of lagrangian equations governed by a velocity-dependent potential energy

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Abstract. It is shown that every generalized force derived from a velocity-dependent potential energy and independent of the acceleration, can be written as an n dimensional 'Lorentz force' with quantities $\boldsymbol{E}$ and $\mathbf{B}$ satisfying generalized Maxwell equations. The equations of motion in an n dimensional cartesian space and with $\mathbf{B} = B(t)\mathbf{B}_0$ are integrated after reduction to canonical form. Expressions for two sets of invariants of the system are constructed and a relationship is shown with a class of well known exact and adiabatic invariants for the motion of a time-dependent harmonic oscillator and of a charged particle in a uniform time-dependent magnetic field.

1. Introduction

It is a well known fact (Goldstein 1959) that the nonhomogeneous Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1,\ldots,n),$$

where q_i are the generalized coordinates, Q_i the generalized forces and T is the kinetic energy, can be made homogeneous, provided the generalized forces are derived from a 'velocity-dependent potential energy' $U(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ by the following prescription:

$$\begin{aligned} Q_i &= -\frac{\partial U}{\partial q_i}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}_i}\right) \\ &= -\frac{\partial U}{\partial q_i}+\frac{\partial^2 U}{\partial \dot{q}_i\partial t}+\sum_{j=1}^{n}\frac{\partial^2 U}{\partial \dot{q}_i\partial q_j}\dot{q}_j+\sum_{j=1}^{n}\frac{\partial^2 U}{\partial \dot{q}_i\partial \dot{q}_j}\ddot{q}_j. \end{aligned} \tag{1}$$

In such case the Lagrange equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i} = 0, \qquad \text{with } L = T-U.$$

Asssuming now that Q_i may not depend upon the generalized acceleration, U must be a linear function of the generalized velocity (Mercier 1963)

$$U = \phi(\boldsymbol{q}, t)-\sum_{j=1}^{n}\dot{q}_jA_j(\boldsymbol{q}, t) = \phi(\boldsymbol{q}, t)-\dot{\boldsymbol{q}}\cdot\boldsymbol{A}(\boldsymbol{q}, t). \tag{2}$$

With this expression for U, (1) becomes

$$Q_i = -\frac{\partial \phi}{\partial q_i} - \frac{\partial A_i}{\partial t} + \sum_{j=1}^{n} \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right) \dot{q}_j,$$

or written in vector notation

$$\boldsymbol{Q} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} + (\nabla \boldsymbol{A} - (\nabla \boldsymbol{A})^{\mathrm{T}}) \cdot \dot{\boldsymbol{q}}. \tag{3}$$

If we define the following quantities:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{4a}$$

$$\mathbf{B} = \nabla \boldsymbol{A} - (\nabla \boldsymbol{A})^{\mathrm{T}}. \tag{4b}$$

$\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \boldsymbol{E} + \mathbf{B} \cdot \dot{\boldsymbol{q}}. \tag{5}$$

Although the quantities $\boldsymbol{E}$ and $\mathbf{B}$ need not describe an electromagnetic phenomenon, equation (5) is a direct generalization of the Lorentz force formula (for unit charge) in three-dimensional electrodynamics. It is easily proved by the lemma of Poincaré for differential forms (eg Flanders 1963), that the necessary and sufficient conditions for the existence of appropriate 'scalar and vector potentials' ϕ and $\boldsymbol{A}$ satisfying (4), are

$$\nabla \boldsymbol{E} - (\nabla \boldsymbol{E})^{\mathrm{T}} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6a}$$

$$\frac{\partial B_{jk}}{\partial q_i} + \frac{\partial B_{ki}}{\partial q_j} + \frac{\partial B_{ij}}{\partial q_k} = 0, \qquad (i, j, k = 1, \ldots, n) \tag{6b}$$

$$\mathbf{B}^{\mathrm{T}} = -\mathbf{B}. \tag{6c}$$

Equations (6*a*) and (6*b*) generalize the homogeneous Maxwell equations.

If $\boldsymbol{q}$ is the vector with components $q_1, \ldots, q_n$ defined in n dimensional cartesian configuration space, the kinetic energy T can be written as $T = \frac{1}{2}\dot{\boldsymbol{q}}^2$. Thus the Lagrangian becomes

$$L = T - U = \tfrac{1}{2}\dot{\boldsymbol{q}}^2 + \dot{\boldsymbol{q}} \cdot \boldsymbol{A} - \phi, \tag{7}$$

which leads to the equations of motion

$$\ddot{\boldsymbol{q}} = \boldsymbol{E} + \mathbf{B} \cdot \dot{\boldsymbol{q}}. \tag{8}$$

In this paper we discuss the case where

$$\phi = 0, \qquad \boldsymbol{A} = \tfrac{1}{2}B(t)\boldsymbol{q} \cdot \mathbf{B}_0, \tag{9}\dagger$$

$\mathbf{B}_0$ being an arbitrary constant skew-symmetric tensor of second rank. The 'generalized fields' $\boldsymbol{E}$ and $\mathbf{B}$ are then

$$\boldsymbol{E} = \tfrac{1}{2}\dot{B}(t)\mathbf{B}_0 \cdot \boldsymbol{q}, \tag{10a}$$

$$\mathbf{B} = B(t)\mathbf{B}_0, \tag{10b}$$

† A scalar potential of the form $\phi = \frac{1}{2}\eta(t)\boldsymbol{q}^2$ could be included without significant modifications.

so that the equations of motion finally become

$$\ddot{\boldsymbol{q}} - B(t)\mathbf{B}_0 \cdot \dot{\boldsymbol{q}} - \tfrac{1}{2}\dot{B}(t)\mathbf{B}_0 \cdot \boldsymbol{q} = 0. \tag{11}$$

In § 2 these equations are brought into a canonical form in order to find the general solution with the aid of the eigenvectors of $\mathbf{B}_0^2$ (§ 3). Typical invariants of the system are easily obtained in § 4. Finally, these invariants are interpreted in § 5 by showing the relationship with adiabatic invariants for the motion of a time-dependent harmonic oscillator and of a charged particle in a uniform time-dependent magnetic field.

2. Reduction of equation (11) to the canonical form

Several authors (eg Hertweck and Schlüter 1957, Lewis 1968b) have established the relationship of the motion of a charged particle with the time-dependent harmonic oscillator. We will demonstrate this fact in our n dimensional case.

If we put

$$q = \mathbf{A} \cdot \boldsymbol{u} \tag{12}$$

into (11) and require the coefficient of $\dot{\boldsymbol{u}}$ in the resulting equation to vanish, the tensor $\mathbf{A}$ must obey the differential equation

$$\dot{\mathbf{A}} - \tfrac{1}{2}B\mathbf{B}_0 \cdot \mathbf{A} = 0.$$

A solution of the latter is given by

$$\mathbf{A} = \exp(\phi(t)\mathbf{B}_0) = \sum_{k=0}^{\infty} \frac{\phi^k}{k!}\mathbf{B}_0^k, \tag{13}$$

where

$$\phi(t) = \tfrac{1}{2}\int^t B(t')\,\mathrm{d}t'. \tag{14}$$

Equations (12) and (13) reduce equations (11) to the canonical form

$$\ddot{\boldsymbol{u}} + \boldsymbol{\Omega}^2 \cdot \boldsymbol{u} = 0, \tag{15}$$

with

$$\boldsymbol{\Omega}^2 = -\tfrac{1}{4}B^2\mathbf{B}_0^2 = -\omega^2(t)\mathbf{B}_0^2. \tag{16}$$

3. Solution of the system (15)

With the idea of decoupling the system of equations (15), we first make some remarks about the eigenvalues and eigenvectors of the skew-symmetric tensor $\mathbf{B}_0$ and the symmetric tensor $\mathbf{B}_0^2$. We first discuss the case for which the number of degrees of freedom is even, $n = 2m$. The case of odd n can easily be reduced to the former one.

$\mathbf{B}_0$ has n constant, purely imaginary, complex conjugate eigenvalues λ_j, λ_j^* ($j = 1, \ldots, m$), which for the moment are supposed to be distinct, with corresponding eigenvectors $\boldsymbol{\zeta}_j$, $\boldsymbol{\zeta}_j^*$. If we put $\boldsymbol{\zeta}_j = \boldsymbol{\xi}_j + \mathrm{i}\boldsymbol{\eta}_j$ ($j = 1, \ldots, m$) for the eigenvector corresponding to the eigenvalue $\lambda_j = \mathrm{i}\omega_j$, we have, equating the real parts and the imaginary parts on both sides of the eigenvalue equation for $\mathbf{B}_0$,

$$\mathbf{B}_0 \cdot \boldsymbol{\xi}_j = -\omega_j \boldsymbol{\eta}_j, \tag{17a}$$

$$\mathbf{B}_0 \cdot \boldsymbol{\eta}_j = \omega_j \boldsymbol{\xi}_j, \tag{17b}$$

from which we get

$$\mathbf{B}_0^2 \cdot \boldsymbol{\xi}_j = -\omega_j^2 \boldsymbol{\xi}_j, \tag{18a}$$

$$\mathbf{B}_0^2 \cdot \boldsymbol{\eta}_j = -\omega_j^2 \boldsymbol{\eta}_j. \tag{18b}$$

Hence, $\boldsymbol{\xi}_j$ and $\boldsymbol{\eta}_j$ are different (real) eigenvectors corresponding to the same (real) eigenvalue $-\omega_j^2$ of $\mathbf{B}_0^2$. The vectors $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ form a complete set of orthonormal eigenvectors, which we suppose to be normalized

$$\boldsymbol{\xi}_j \cdot \boldsymbol{\xi}_k = \boldsymbol{\eta}_j \cdot \boldsymbol{\eta}_k = \delta_{jk}, \qquad \boldsymbol{\xi}_j \cdot \boldsymbol{\eta}_k = 0, \qquad (j, k = 1, \ldots, m). \tag{19}$$

We now shall attempt to find a particular solution of (15) of the form

$$\boldsymbol{u}_j = w_j(t) \exp(\pm \mathrm{i}\psi_j(t)) \boldsymbol{v}_j, \tag{20}$$

where $\boldsymbol{v}_j$ is any eigenvector $\boldsymbol{\xi}_j$ or $\boldsymbol{\eta}_j$ of $\mathbf{B}_0^2$. If we choose

$$\psi_j(t) = \int^t \frac{\mathrm{d}t'}{w_j^2} \tag{21}$$

$\boldsymbol{u}_j$ will be a particular solution of (15), provided $w_j(t)$ is a particular solution of the non-linear differential equation

$$\ddot{w}_j + \omega_j^2 \omega^2(t) w_j - w_j^{-3} = 0. \tag{22}$$

We then immediately conclude that the general solution of equation (15) can be written as

$$\boldsymbol{u} = \sum_{j=1}^{m} w_j(t) \{(A_j \boldsymbol{\xi}_j + B_j \boldsymbol{\eta}_j) \cos \psi_j + (C_j \boldsymbol{\xi}_j + D_j \boldsymbol{\eta}_j) \sin \psi_j\}, \tag{23}$$

A_j, B_j, C_j, D_j being $2n$ arbitrary constants. It is easily verified that the $2n$ terms contained in (23) are indeed linearly independent particular solutions of (15). Equations (12), (13) and (23) then yield the general solution for $\boldsymbol{q}$, which after some straightforward calculations, using equations (17) and (18), can be written

$$\begin{aligned}
\boldsymbol{q} &= \exp(\phi \mathbf{B}_0) \cdot \boldsymbol{u} \\
&= \sum_{j=1}^{m} w_j \cos \psi_j \{(A_j \cos(\omega_j \phi) + B_j \sin(\omega_j \phi)) \boldsymbol{\xi}_j - (A_j \sin(\omega_j \phi) - B_j \cos(\omega_j \phi)) \boldsymbol{\eta}_j\} \\
&\quad + \sum_{j=1}^{m} w_j \sin \psi_j \{(C_j \cos(\omega_j \phi) + D_j \sin(\omega_j \phi)) \boldsymbol{\xi}_j \\
&\quad - (C_j \sin(\omega_j \phi) - D_j \cos(\omega_j \phi)) \boldsymbol{\eta}_j\}.
\end{aligned} \tag{24}$$

If $n = 2m+1$, the matrix associated with the tensor $\mathbf{B}_0$ is singular and has at least one vanishing eigenvalue $\lambda_0 = 0$. The equations (23) and (24) then must be augmented with an independent solution $(A_0 t + B_0)\boldsymbol{v}_0$, where $\boldsymbol{v}_0$ is the eigenvector corresponding to λ_0.

If the multiplicity of an eigenvalue of $\mathbf{B}_0$ is greater than one, we still can construct a set of orthonormal eigenvectors for the tensor $\mathbf{B}_0^2$. The independency of the particular solutions in (23) remains valid, even though we only need one $w_j(t)$ for each degenerate eigenvalue.

The equations (22) ($j = 1, \ldots, m$) are generalizations of similar equations obtained by different authors (Courant and Snyder 1958, Lewis 1968a, Symon 1970) in one- and two-dimensional problems. It was, however, never mentioned before, that in order to construct a particular solution of the nonlinear equation (22), it suffices to find a particular solution of the linear equation

$$\ddot{x}_j + \omega_j^2 \omega^2(t) x_j = 0.$$

Indeed, if x_j is any such particular solution, then

$$w_j = x_j \left\{ 1 + \left(\int^t x_j^{-2} \, \mathrm{d}t' \right)^2 \right\}^{1/2}$$

is a corresponding particular solution of equation (22).

4. Invariants of the system

From the general solution (23) for $\boldsymbol{u}$ and the orthogonality equations (19), the following relations may be established:

$$w_k \boldsymbol{\xi}_k \cdot \dot{\boldsymbol{u}} - \dot{w}_k \boldsymbol{\xi}_k \cdot \boldsymbol{u} = C_k \cos \psi_k - A_k \sin \psi_k$$

and

$$w_k^{-1} \boldsymbol{\xi}_k \cdot \boldsymbol{u} = C_k \sin \psi_k + A_k \cos \psi_k,$$

from which m invariants can immediately be obtained, namely

$$2I_k = (w_k \boldsymbol{\xi}_k \cdot \dot{\boldsymbol{u}} - \dot{w}_k \boldsymbol{\xi}_k \cdot \boldsymbol{u})^2 + (w_k^{-1} \boldsymbol{\xi}_k \cdot \boldsymbol{u})^2 = A_k^2 + C_k^2, \qquad (k = 1, \ldots, m). \tag{25a}$$

In the same way, it is seen that

$$2I_{k+m} = (w_k \boldsymbol{\eta}_k \cdot \dot{\boldsymbol{u}} - \dot{w}_k \boldsymbol{\eta}_k \cdot \boldsymbol{u})^2 + (w_k^{-1} \boldsymbol{\eta}_k \cdot \boldsymbol{u})^2 = B_k^2 + D_k^2 \qquad (k = 1, \ldots, m) \tag{25b}$$

are m other invariants. With the help of the transformation (12), (13), these $n (= 2m)$ invariants I_k, I_{m+k} can be expressed in terms of the original variables $\boldsymbol{q}$. If we consider the components of $\boldsymbol{q}$ with respect to the n orthogonal unit vectors $\boldsymbol{\xi}_k$, $\boldsymbol{\eta}_k$, defined by

$$\boldsymbol{q} \cdot \boldsymbol{\xi}_k = \bar{q}_k, \qquad \boldsymbol{q} \cdot \boldsymbol{\eta}_k = \bar{q}_{m+k} \qquad (k = 1, \ldots, m) \tag{26}$$

and use equations (17), (18), (19), the invariants I_k, I_{m+k} take the form

$$2I_k = \left(w_k^2 \frac{\mathrm{d}}{\mathrm{d}t} \{ w_k^{-1} (\bar{q}_k \cos(\omega_k \phi) - \bar{q}_{m+k} \sin(\omega_k \phi)) \} \right)^2 + \{ w_k^{-1} (\bar{q}_k \cos(\omega_k \phi) - \bar{q}_{m+k} \sin(\omega_k \phi)) \}^2, \qquad (k = 1, \ldots, m) \tag{27a}$$

$$2I_{m+k} = \left(w_k^2 \frac{\mathrm{d}}{\mathrm{d}t} \{ w_k^{-1} (\bar{q}_{m+k} \cos(\omega_k \phi) + \bar{q}_k \sin(\omega_k \phi)) \} \right)^2 + \{ w_k^{-1} (\bar{q}_{m+k} \cos(\omega_k \phi) + \bar{q}_k \sin(\omega_k \phi)) \}^2, \qquad (k = 1, \ldots, m). \tag{27b}$$

Another set of m invariants is related to the existence of a cyclic angle variable in each $(\bar{q}_k, \bar{q}_{m+k})$ plane. According to equations (7) and (9), the lagrangian of the system (11) is given by

$$L = \tfrac{1}{2}\dot{\boldsymbol{q}}^2 + \omega(t)\boldsymbol{q} \cdot \mathbf{B}_0 \cdot \dot{\boldsymbol{q}}$$

$$= \tfrac{1}{2}\sum_{k=1}^{m} (\dot{\bar{q}}_k^2 + \dot{\bar{q}}_{m+k}^2) + \omega(t)\sum_{k=1}^{m} \omega_k(\bar{q}_k\dot{\bar{q}}_{m+k} - \bar{q}_{m+k}\dot{\bar{q}}_k), \tag{28}$$

or, introducing polar coordinates in each $(\bar{q}_k, \bar{q}_{m+k})$ plane,

$$L = \tfrac{1}{2}\sum_{k=1}^{m} (\dot{\rho}_k^2 + \rho_k^2\dot{\theta}_k^2) + \omega(t)\sum_{k=1}^{m} \omega_k\rho_k^2\dot{\theta}_k,$$

where $\bar{q}_k = \rho_k \cos\theta_k$ and $\bar{q}_{m+k} = \rho_k \sin\theta_k$. Each θ_k being cyclic, we conclude that the conjugate momenta

$$p_{\theta_k} = \frac{\partial L}{\partial \dot{\theta}_k} = \rho_k^2(\dot{\theta}_k + \omega_k\omega(t)) \qquad (k = 1, \ldots, m) \tag{29}$$

are m constants of motion.

5. Interpretation of the invariants

Making the scalar product of equations (11) once with $\boldsymbol{\xi}_k$, once with $\boldsymbol{\eta}_k$, we obtain the system of equations of motion for $\bar{q}_k$ and $\bar{q}_{m+k}$,

$$\ddot{\bar{q}}_k - 2\omega_k\omega(t)\dot{\bar{q}}_{m+k} - \omega_k\dot{\omega}(t)\bar{q}_{m+k} = 0, \tag{30a}$$

$$\ddot{\bar{q}}_{m+k} + 2\omega_k\omega(t)\dot{\bar{q}}_k + \omega_k\dot{\omega}(t)\bar{q}_k = 0. \tag{30b}$$

Since for each k, the equations (30) do not involve components of $\boldsymbol{q}$ other than $\bar{q}_k$ and $\bar{q}_{m+k}$, we can treat them separately. They could, for example, describe the projection of the motion of a charged particle in a time-varying uniform magnetic field on a plane perpendicular to the magnetic field direction. Then $2\omega_k\omega(t)$ stands for $B_k(t)$, the absolute value of the magnetic field.

Lewis (1968b) has studied these equations introducing a complex variable. Reducing the resulting equation to the canonical form (the one-dimensional harmonic oscillator), he obtained a complex invariant, which is not very significant for the charged particle motion.

We prefer to reduce the equations (30) directly to the canonical form by means of the transformation (12), (13), which here takes the form

$$\bar{q}_k = \bar{u}_k \cos(\omega_k\phi) + \bar{u}_{m+k}\sin(\omega_k\phi), \tag{31a}$$

$$\bar{q}_{m+k} = -\bar{u}_k \sin(\omega_k\phi) + \bar{u}_{m+k}\cos(\omega_k\phi), \tag{31b}$$

with $\bar{u}_k = \boldsymbol{u} \cdot \boldsymbol{\xi}_k$ and $\bar{u}_{m+k} = \boldsymbol{u} \cdot \boldsymbol{\eta}_k$. Geometrically this represents a transition to a rotating reference frame and reduces equations (30) to those of a two-dimensional harmonic oscillator

$$\ddot{\bar{u}}_k + \omega_k^2\omega^2(t)\bar{u}_k = 0, \tag{32a}$$

$$\ddot{\bar{u}}_{m+k} + \omega_k^2\omega^2(t)\bar{u}_{m+k} = 0. \tag{32b}$$

So the invariants I_k and I_{m+k} given by equations (25), which were obtained directly from the general solution of equations (11), are in agreement with the results obtained by Lewis (1968a) for the one-dimensional oscillator.

Furthermore, if $\omega(t) = \frac{1}{2}B(t)$ is a slowly varying function of time, and if a particular solution for w_k is obtained by a series expansion in some small parameter ϵ, then I_k and I_{m+k} give rise to so called 'asymptotic or adiabatic' invariants to all orders of the system (32), the zeroth order terms corresponding exactly with the historically first concept of adiabatic invariance (Burgers 1917).

It is easily verified that the invariants I_k, I_{m+k} in the form (27), for slowly varying $\omega(t)$ and using the series solution for w_k (Lewis 1968a), still have the property which nowadays is mostly used to define adiabatic invariance to all orders (eg Coffey 1966, Stern 1971), namely

$$\dot{I}_\alpha^{(n)} = O(\epsilon^{n+1}), \qquad (\alpha = k \text{ or } m+k)$$

where $I_\alpha^{(n)}$ stands for the sum of the first $n+1$ terms in the expansion of I_α as a power series in ϵ. The same is true of course for the invariant

$$J_k = \tfrac{1}{2}(I_k + I_{m+k} - p_{\theta_k}), \tag{33}$$

p_{θ_k} being always a time-independent invariant of the system (30) in hamiltonian form.

Now a straightforward calculation shows that the zeroth order term in the expansion for J_k equals,

$$J_k^{(0)} = \frac{\dot{\bar{q}}_k^2 + \dot{\bar{q}}_{m+k}^2}{4\omega_k\omega(t)} = \frac{\dot{\bar{q}}_k^2 + \dot{\bar{q}}_{m+k}^2}{2B_k(t)} = \frac{v_\perp^2}{2B_k} \tag{34}$$

so that J_k yields the adiabatic invariant series for the magnetic moment of the charged particle motion governed by equations (30).

Note that, applying on the system (30) a perturbation technique, established by Kruskal (1962), one obtains an invariant series with of course the same zeroth order term. The first order term obtained in this way after rather tedious calculations still equals the first order term in the expansion for J_k, which is merely generated by $\frac{1}{2}(I_k + I_{m+k})$. In particular, the result (34) shows that the adiabatic invariance of the magnetic moment in the motion of a charged particle in a uniform but time-dependent magnetic field is essentially the same as the adiabatic invariance of the ratio E/v for the harmonic oscillator, a connection already mentioned by Chandrasekhar (1958).

We finally remark that, if the invariant J_k is written in polar coordinates, we get

$$J_k = \frac{1}{4}(w_k^2\dot{R}_k)^2 + \frac{1}{4}\left(\frac{p_{\theta_k}}{R_k} - R_k\right)^2, \qquad \text{with } R_k = \frac{\rho_k}{w_k},$$

while $2(I_k + I_{m+k})$ gives an expression equivalent to the invariant mentioned by Lewis (1968b) in his equation (22).

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